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Centrifugal instabilities in finite containers: a periodic model

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A simplified model problem has recently been suggested by Schaeffer (1980) in order to explain the results obtained by Benjamin (1978) in his investigation of Taylor vortices in short cylinders. In particular Schaeffer reproduces the results obtained by Benjamin for cylinders so short that only two-cell or four-cell flows are possible. The model given by Schaeffer has artificial conditions imposed on the fluid velocity field at the end walls. These conditions depend on a parameter α and reduce to the no-slip condition when $\alpha = 1$. If $\alpha = 0$ the conditions require that the normal component of the velocity and the normal derivative of the tangential velocity vanish at the ends. In this case the onset of Taylor vortex-like motion occurs as a bifurcation from purely circumferential flow. If α is now taken to be small and positive, there is no bifurcation and the circulatory flow develops smoothly. We shall use perturbations method for the case of small α. The imperfect bifurcation problem which we obtain predicts some results consistent with those of Benjamin.

1. Introduction

In recent years there have been several investigations of the role of end effects in hydrodynamic stability theory. In particular Benjamin (1978) has investigated the classical Taylor vortex problem in short cylinders. Earlier investigations of end effects in connexion with the Taylor problem had concentrated on longer cylinders (see, for example, Cole 1976).

Benjamin discussed the experimental results which he obtained for flows in cylinders having the outer cylinder and the end walls fixed. He also discussed the connexion between these results and some generic properties of such flows which he predicted. In particular Benjamin investigated the question of how many cells can be observed in a given experimental configuration at a fixed Reynolds number. Most of the results given were for cylinders so short that only two or four cells could be accommodated. Some of his experimental results are shown in figure 1 where R is the Reynolds number of the flow and L is the non-dimensional length of the cylinders.

Benjamin found that if the Reynolds number R is slowly increased from zero, then the initial flow has two cells if $L < L_1^*$ and four cells if $L < L_2^*$. Moreover, this flow develops smoothly when R is increased until wavy modes occur as secondary bifurcations from the primary flow for $R \sim 10^3$. If the point (L, R) lies to the right of the curves I or II, then secondary modes with two-cell or four-cell flows are possible depending on whether $L > L_2^*$ or $L < L_1^*$. Such flows cannot be obtained by increasing

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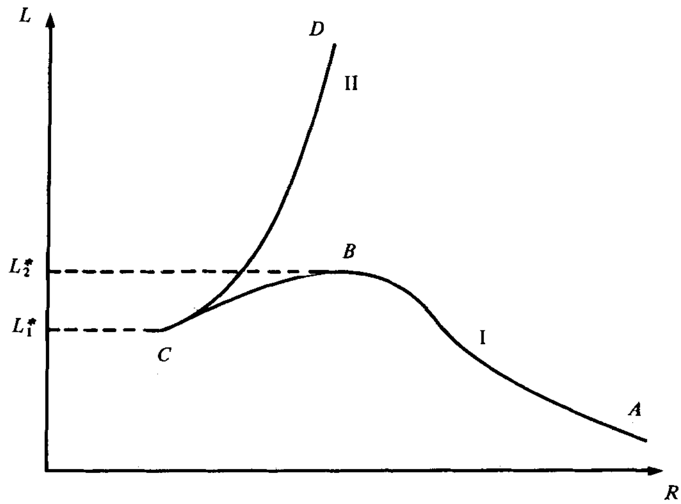

FIGURE 1. Results from Benjamin's (1978) experiments.

the Reynolds number slowly from zero. Now suppose that L lies in the range (L_1^*, L_2^*) and the Reynolds number is again slowly increased from zero. The primary flow now loses its stability to a secondary axisymmetric mode when CB is crossed. However, if, after the latter mode has been established, the Reynolds number is decreased the flow does not jump back to the primary flow until CD is crossed so that the flow exhibits a hysteresis phenomenon. We further note that another stable secondary flow is possible if (L, R) lies to the right of I with $L_1^* < L < L_2^*$. This secondary flow has four cells and is described by Benjamin as a stronger version of the primary flow possible for (L, R) lying just to the left of CB.

In addition to the above symmetric modes, Benjamin found secondary modes with an odd number of cells. In recent papers, Blennerhassett & Hall (1979) and Hall (1980) have discussed the linear and nonlinear stability of a simplified model of Benjamin's experiments. The model allowed the end walls to rotate in a manner such that the circulatory part of the primary flow driven by the ends was weak. This was done by specifying that at the end walls the fluid velocity in the azimuthal direction was close to the value if would have if the cylinders were infinite. The other velocity components were set equal to zero at the end walls. It was found by Blennerhassett & Hall that, at a given value of L, an infinite countable set of possible flows could exist, each having a different number of cells which could be odd or even. In general, the Taylor numbers at which these linear modes first occur are different but, for certain values or L, the Taylor numbers of the first odd and even modes coincide. A nonlinear calculation to determine the preferred flow in a neighbourhood of such points was given by Hall. It was found that the primary flow always had an even number of cells but that, in certain cases, secondary bifurcations to flows with an odd number of cells could occur. Hall also showed how, by taking the limit $L \to \infty$, results consistent with those for the infinite problem are obtained.

Schaeffer (1980) has given an alternative simplified model of the experiments of Benjamin in an attempt to reproduce the results of figure 1. Schaeffer assumed that u, v, w, the radial, azimuthal, and axial velocities of the flow, satisfied the following conditions at the end walls

$$(1-\alpha)\frac{\partial u}{\partial n}+\alpha u = 0, \quad (1-\alpha)\frac{\partial v}{\partial n}+\alpha v = 0, \quad w = 0, \tag{1.1a,b,c}$$

where α is a homotopy parameter satisfying $0 \leqslant \alpha \leqslant 1$ and $\partial/\partial n$ denotes the normal derivatives.

If we set $\alpha = 1$ the above conditions reduce to the no-slip condition appropriate to Benjamin's results. If $\alpha = 0$, we obtain a simpler problem which leads to a linear stability problem for a finite region which we solve by using a single solution of the corresponding infinite problem. Schaeffer argues that by then obtaining results for the case $\alpha > 0$, $\alpha \ll 1$, we can infer results for the case $\alpha = 1$.

Using singularity theory, Schaeffer argues that the results of figure 1 are plausible. However, as stated by Schaeffer, the analysis he gave is valid only for the $2m$, $2m+2$ cell interaction problem with $m \geqslant 2$. Moreover, the coefficients in the bifurcation equations obtained by Schaeffer had to have certain properties in order that the equations could describe the results of figure 1. The actual values of the coefficients were not calculated by Schaeffer.

In this paper we shall investigate the model proposed by Schaeffer but applied to the 2- and 4-cell interaction problem. This will be done by asymptotic methods. We shall see that the bifurcation equations investigated by Schaeffer are quite different than those appropriate to this problem. We shall show that, in sufficiently long cylinders the primary flow develops smoothly and is a four-cell flow, whilst the same result holds for a two-cell primary flow in sufficiently short cylinders. We shall also use our nonlinear calculation to generate curves corresponding to the curves BC and BA of figure 1; we are unable to predict the phenomenon associated with the curve CD in figure 1. The procedure adopted is as follows. In § 2 we shall formulate the stability problem for the model of Schaeffer but applied to the so-called narrow gap limit. In § 3 we shall find the possible flows in a region of order $\alpha^{\frac{1}{2}}$ around a point corresponding to B in figure 1. Some of these solutions break down in a region of order $\alpha^{\frac{2}{3}}$ near the curve BA. The development of these solutions in this region is determined in § 4. In § 5 we investigate the flows possible when (R, L) differs by an amount of $O(1)$ from the co-ordinates of the point B in figure 1. Some discussion of the wide gap problem is also given in § 5, and our conclusions are summarized in § 6.

2. Formulation of the problem

We investigate the flow of an incompressible fluid of kinematic viscosity ν between concentric circular cylinders of length $2Ld$ and radii R_0 and R_0+d. The inner cylinder rotates with angular velocity Ω whilst the outer cylinder is held fixed. We shall assume that $d/R_0 \ll 1$ so that the small gap approximation can be made.

We follow Hall (1979) and define dimensionless variables x and ϕ by

$$x = (r-R_0)/d, \quad \phi = z/d, \tag{2.1a,b}$$

where (r, θ, z) are cylindrical polar co-ordinates with $r = 0$ and $z = 0$ corresponding to the axis and the plane midway between the ends of the cylinders respectively. The cylinders are taken to be rigid so that the velocity of the fluid satisfies

$$(u, v, w) = (0, \Omega R_0, 0), \quad x = 0. \tag{2.2}$$
$$(u, v, w) = (0, 0, 0), \quad x = 1.$$

Following, Schaeffer, we now impose the conditions

$$(1-\alpha)\frac{\partial u}{\partial n}+\alpha u = 0, \quad (1-\alpha)\frac{\partial v}{\partial n}+\alpha v = 0, \quad w = 0 \quad \text{on} \quad \phi = \pm L. \quad (2.3a,b,c)$$

Here $\partial/\partial n$ denotes the normal derivative whilst α is a fixed constant. If $\alpha = 1$ the conditions (2.3) reduce to the no-slip conditions appropriate to the experiments of Benjamin whilst if $\alpha = 0$ they reduce to

$$w = 0, \quad \frac{\partial u}{\partial n} = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \phi = \pm L. \quad (2.4a,b,c)$$

The boundary conditions enable us to make some analytical progress in determining the flow between the cylinders. If we now choose α such that $0 < \alpha \ll 1$, then we can at least hope that the results obtained for the perturbed problem shed some light on the problem with $\alpha = 1$. We note that the conditions (2.4) lead to a basic flow

$$(u, v, w) = \Omega R_0(0, 1-x, 0), \quad (2.5)$$

which is just the basic circumferential flow of the corresponding infinite problem.

We define a Taylor number T by

$$T = 2\Omega_0^2 R_0 d^3/\nu^2, \quad (2.6)$$

and *a* time variable τ^* by

$$\tau^* = \frac{\nu}{d^2}t, \quad (2.7)$$

For the moment, we assume that $\alpha = 0$ and perturb the basic flow axisymmetrically such that the disturbed velocity field is

$$\left(\frac{-\nu}{2d}u, \quad \Omega R_0(1-x)+\frac{\Omega R_0}{2}v, \quad \frac{-\nu}{2d}w\right).$$

We can show from the momentum and the continuity equations by performing the usual manipulations that u, v, w satisfy

$$\left\{l-\frac{\partial}{\partial \tau^*}\right\} lu - T(1-x)v_{\phi\phi} = -\tfrac{1}{2}Q_{1\phi\phi}+\tfrac{1}{2}Q_{2x\phi}, \quad (2.8a)$$

$$\left\{l-\frac{\partial}{\partial \tau^*}\right\} v-u = -\tfrac{1}{2}Q_3, \quad u_x+w_\phi = 0, \quad (2.8b)$$

$$u_x+w_\phi = 0, \quad (2.8c)$$

where the operator l is defined by

$$l = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2},$$

and $$Q_1 = uu_x+wu_\phi-\tfrac{1}{2}Tv^2, \quad Q_2 = uw_x+ww_\phi, \quad Q_3 = uv_x+wv_\phi. \quad (2.9a,b,c)$$

Following Schaeffer, we consider flows which are symmetric about the plane $z = 0$. If we neglect the nonlinear terms in (2.8) and write

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u_n(x)\cos k_n\phi \\ v_n(x)\cos k_{n\phi} \\ -\dfrac{u_n'(x)}{k_n}\sin k_n\phi \end{pmatrix}, \quad (2.10)$$

then (u_n, v_n) is determined by the sixth-order ordinary differential system

$$\left.\begin{aligned}\left[\frac{d^2}{dx^2}-k_n^2\right]^2 u_n + Tk_n^2(1-x)\,v_n = 0,\\ \left[\frac{d^2}{dx^2}-k_n^2\right] v_n - u_n = 0, \quad u_n = v_n = u_n' = 0, \quad x = 0, 1.\end{aligned}\right\} \qquad (2.11)$$

For a given value of T, there is an infinite set $\{k_n\}$ of eigenvalues of the above system. [See Blennerhassett & Hall (1979) for a more detailed discussion of these eigenvalues.] If T is greater than its critical value T_c of the infinite problem, then there will be at least four real wavenumbers $\pm k_1$, $\pm k_2$ and we choose to order them such that $0 < k_1 < k_2$. The remaining eigenvalues are ordered on the basis of the nature of the corresponding eigenfunctions in the manner suggested by Blennerhassett & Hall.

The disturbance given by (2.10) must now be made to satisfy the end conditions given by (2.4). However the reason for the choice of end conditions now becomes apparent, since (2.4) is satisfied if

$$k_n L = m\pi, \quad m = 1, 2, 3, \ldots, \qquad (2.12)$$

and the flow field then has $2m$ cells in $(-L, L)$. Thus the conditions (2.4) are satisfied by a single axial mode in contrast to Blennerhassett & Hall where all the axial modes were required to satisfy the end conditions. Since the wavenumber k_n depends on T, it follows that (2.12) specifies an eigenrelation $T = T_{mn} = T_{mn}(L)$. At a point of intersection of the eigencurves $T_{mn} = T_{mn}(L)$, $T_{ij} = T_{ij}(L)$, two possible types of disturbance with $2m$ and $2i$ cells respectively exist. Our concern here is with the point of intersection of the m 1 and $m = 2$ curves obtained from the eignenvalues k_1 and k_2. (Note that the curves $T_{ml} = T_{ml}(L)$, $T_{m2} = T_{m2}(L)$ join smoothly when $T = T_c$.) This point of intersection corresponds to the value of the Taylor number at which

$$k_2 = 2k_1,$$

and by solving (2.11) numerically we find that in this case

$$k_1 = k = 2{\cdot}17, \quad k_2 = 2k, \quad T = T^* = 4010. \qquad (2.13\,a,b,c)$$

The corresponding value of L is then given by

$$L = L^* = 1{\cdot}45. \qquad (2.14)$$

For convenience we shall denote the Taylor numbers corresponding to the two-cell and four-cell modes by $\hat{T}_2(L)$ and $\hat{T}_4(L)$ respectively. These curves are shown in figure 2 and we note that, in each case, the parts of the curves lying to the right and the left of the minimum point correspond to the wavenumbers k_1 and k_2 respectively. The curves intersect at the point J. We note that the two-cell mode is the most dangerous for $L < L^*$, otherwise the four-cell mode is the most dangerous. We shall now construct weakly nonlinear solutions of (2.8) valid in the neighbourhood of J. This nonlinear analysis will enable us to identify J with the point B in figure 1.

3. Weakly nonlinear solutions for $(T-T^*) \sim (L-L^*) \sim \alpha^{\frac{1}{2}}$

We are concerned here with the nonlinear development of the two linear modes with wavenumbers k and $2k$. If these modes interact once with each other then each is

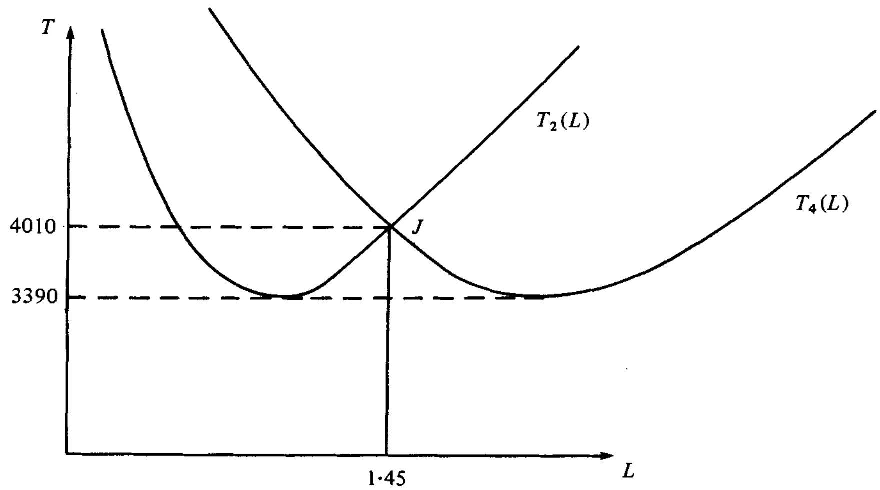

FIGURE 2. The neutral curves for the two-cell and four-cell modes.

reinforced. If $\alpha = 0$ such unforced solutions can be constructed in a neighbourhood of (L^*, T^*). However, we shall determine how these modes can be forced by the end conditions when $\alpha \neq 0$.

Suppose that (T, L) is perturbed by an amount of order α^β from its value (T^*, L^*) at J, the point of intersection of the curves $T = \hat{T}_2(L)$, $T = \hat{T}_4(L)$. Since one interaction of the fundamental modes given in §2 reinforces these modes, we expect a finite amplitude solution of the unforced problem to be $O(\alpha^\beta)$. However if $\alpha \neq O$, the conditions on the disturbance velocity field (u, v, w) at $\phi = \pm L$ become

$$\frac{\partial u}{\partial n} = \alpha\left(\frac{\partial u}{\partial n} - u\right), \quad \frac{\partial v}{\partial n} = -2\alpha(1-x) + \alpha\left(\frac{\partial v}{\partial n} - v\right), \quad w = 0 \quad \text{on} \quad \phi = \pm L, \tag{3.1 a, b, c}$$

so that the forced motion will resonate for $(T - T^*) \sim O(\alpha)^\beta$ and will be of order $\alpha^{1-\beta}$. Thus a balance between the resonating solution and the nonlinear solution is achieved if we choose $\beta = 1/2$. We therefore expand $\mathbf{u} = (u, v, w)$, T, and L in the forms

$$\mathbf{u} = \alpha^{\frac{1}{2}}\mathbf{U}_0 + \alpha\mathbf{U}_1 + \alpha^{\frac{3}{2}}\mathbf{U}_2 + \ldots, \tag{3.2a}$$

$$T = T^* + \alpha^{\frac{1}{2}}T_1 + \alpha T_2 + \ldots, \tag{3.2b}$$

$$L = L^* + \alpha^{\frac{1}{2}}L_1 + aL_2 + \ldots, \tag{3.2c}$$

and define a slow time scale τ by

$$\tau = \alpha^{\frac{1}{2}}\tau^*. \tag{3.3}$$

We assume that the coefficients in the expansions (3.2b, c) are given together with α and this specifies L and T. The coefficients in the expansion (3.2a) are functions of ϕ, α and τ which we can obtain by equating like powers of $\alpha^{\frac{1}{2}}$ in (2.8) after substituting for $\mathbf{u}$, T from (3.2a, b) and replacing $\partial/\partial\tau^*$ by $\alpha^{\frac{1}{2}}\,\partial/\partial\tau$. At order $\alpha^{\frac{1}{2}}$ this procedure leads to a linear partial differential system for $\mathbf{U}_0$ which, when solved subject to the appropriate boundary conditions, gives

$$\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix} = A(\tau)\begin{pmatrix} \cos k\phi u_1 \\ \cos k\phi v_1 \\ -\dfrac{\sin k\phi}{k} u_1' \end{pmatrix} + B(\tau)\begin{pmatrix} \cos 2k\phi u_2 \\ \cos 2k\phi v_2 \\ -\dfrac{\sin 2k\phi}{2k} u_1' \end{pmatrix}, \tag{3.4}$$

where (u_1, v_1), (u_2, v_2) are the eigenfunctions associated with the wavenumbers k and $2k$ whilst $A(\tau)$ and $B(\tau)$ are amplitude functions to be determined at higher order.

At order α we obtain a partial differential system for $\mathbf{U}_1$ which, ignoring the end conditions, has the solution

$$\begin{pmatrix} U_1 \\ V_1 \\ W_1 \end{pmatrix} = -\left\{\epsilon_1 \frac{dA}{d\tau} + \delta_1 T_1 A + \gamma_1 AB\right\}\begin{pmatrix} \phi \sin k\phi u_1 \\ \phi \sin k\phi v_1 \\ \dfrac{[k\phi \cos k\phi - \sin k\phi]}{k^2} u_1' \end{pmatrix}$$

$$-\left\{\epsilon_2 \frac{dB}{d\tau} + \delta_2 T_1 B + \gamma_2 A^2\right\}\begin{pmatrix} \phi \sin 2k\phi u_2 \\ \phi \sin 2k\phi v_2 \\ \dfrac{[2k\phi \cos 2k\phi - \sin 2k\phi]}{4k^2} u_2' \end{pmatrix}$$

$$+\sum_{n=3}^{\infty} \mu_n \begin{pmatrix} \cos k_n\phi u_n \\ \cos k_n \phi v_n \\ -\dfrac{\sin k_n\phi}{k_n} u_n' \end{pmatrix} + \begin{Bmatrix} \text{terms proportional to} \\ \sin nk\phi,\ \cos nk\phi,\ n = 1,2,3,4 \\ \text{and terms independent of } \phi \end{Bmatrix}. \tag{3.5}$$

For the sake of brevity we do not give explicit forms for the terms in the curly brackets since they automatically satisfy the required conditions at $\phi = +L$. In fact the ϕ-dependent terms not shown explicitly are determined by inhomogeneous forms of (2.11) with $k_n = nk$ and comprise terms proportional to $dA/d\tau$, $dB/d\tau$, A, B, A^2 and AB. In addition there is an azimuthal mean flow velocity field proportional to A^2, B^2. The constants ϵ_1, ϵ_2, etc. appearing in (3.5) are defined by

$$\epsilon_n = \left\{-\int_0^1 u_n^+[u_n'' - k_n^2 u_n] + v_n^+ v_n \,dx\right\}\lambda_n^{-1}, \quad n = 1, 2, \tag{3.6a}$$

$$\delta_n = \left\{k_n^2 \int_0^1 u_n^+ [1-x] v_n \,dx\right\}\lambda_n^{-1} \quad n = 1, 2, \tag{3.6b}$$

$$\gamma_1 = +\frac{1}{4}\left\{\int_0^1 H u_1^+ + G v_1^+ dx\right\}\lambda_1^{-1}, \quad \gamma_2 = \frac{1}{4}\left\{\int_0^1 C u_2^+ + D v_2^+ \,dx\right\}\lambda_2^{-1}. \tag{3.6c, d}$$

Here (u_n^+, v_n^+) is the function pair adjoint to (u_n, v_n) whilst λ_n, H, G, C, D are given by

$$\lambda_n = \int_0^1 \{2k_n v_n^+ v_n + 4u_n^+[u_n'' - k_n^2 u_n]k_n - 2T^*(1-x) k_n v_n u_n^+\} \,dx, \tag{3.7a}$$

$$H = -k^2\{-T^* v_1 v_2 + \tfrac{3}{2} u_1 u_2' + 3u_1' u_2\} + \frac{u_1'' u_2' - u_1 u_2'''}{2} + u_1''' u_2 - u_1' u_2'', \tag{3.7b}$$

$$G = u_1 v_2' + u_2 v_1' + 2u_1' v_2 + \tfrac{1}{2} u_2' v_1, \tag{3.7c}$$

$$C = 2k^2 T^* v_1^2 + 2(u_1 u_1''' - u_1' u_1''), \tag{3.7d}$$

$$D = +u_1 v_1' - u_1' v_1. \tag{3.7e}$$

The constants $\{\mu_n\}$, $n = 3, 4, \ldots$, must now be chosen so as to satisfy the end conditions. In view of the symmetry of the flow about $\phi = 0$, we need only consider the conditions at $\phi = L$, which from (3.1), (3.2), (3.4), and (3.5), give

$$Xu_1 - 2Yu_2 + \frac{1}{\pi}\sum_{n=3}^{\infty}\mu_n \cos k_n L^* u_n = 0 \tag{3.8a}$$

$$Xv_1 - 2Yv_2 + \frac{1}{\pi}\sum_{n=3}^{\infty}\mu_n \cos k_n L^* v_n = \frac{-2}{\pi}(1-x) \tag{3.8b}$$

$$-Xu_1 + \tfrac{1}{2}Yu_2' + \frac{k^2}{\pi}\sum_{n=3}^{\infty}\frac{\mu_n}{k_n}\sin k_n L^* u_n' = 0. \tag{3.8c}$$

Here the coefficients X and Y are defined by

$$X = \epsilon_1\frac{dA}{d\tau} + \delta_1 AT_1 + \gamma_1 AB_1 + \frac{k^2}{\pi}L_1 A, \tag{3.9a}$$

$$Y = \epsilon_2\frac{dB}{d\tau} + \delta_2 BT_1 + \gamma_2 A^2 + \frac{2k^2}{\pi}L_1 B. \tag{3.9b}$$

The above equations must be satisfied at every value of x in (0, 1). We note that, since $v_n(0) = 0$, we cannot satisfy (3.8b) at $x = 0$ and hence we expect Gibb's phenomenon to occur. The eigenfunction pairs (u_1, v_1) and (u_2, v_2) were normalized such that $u_2'''(1) = u_1'''(1) = 1$, in which case $u_1(x)$, $u_2(x)$ are both negative in (0, 1). The equations (3.8a, b, c) are now multiplied by $\sin m\pi x$, $m = 1, 2, \ldots, M$ and integrated from $x = 0$ to $x = 1$. If we set $\mu_n = 0$ for $n > 3M$, such a procedure gives $3M$ equations for the $3M$ unknown quantities X, Y,$\{\mu_n\}$, $3 \leqslant n \leqslant M$. If the eigenvalues are ordered in the manner suggested by Blennerhassett & Hall, such a procedure gives values for X and Y which converge quickly when M increases. The results obtained by taking $M = 2, 4, \ldots, 10$ are shown in table 1. We note that the coefficients X, Y, etc. could also be obtained by collocation methods but, as found by Stewartson & Weinstein (1979), the discontinuity at $x = 0$, leads to an oscillation in the value of (X, Y) when more collocation points are used.

The remaining coefficients in (3.8) were also calculated numerically and the results obtained are shown in table 2. For convenience, using the values given in table 1 (for $M = 10$) and table 2, we can write the amplitude equations for A and B in the form

$$\frac{dA}{d\tau} = \sigma_1 A\{T_1 - \zeta_1 L_1\} + e_1 AB + f_1, \tag{3.10a}$$

$$\frac{dB}{d\tau} = \sigma_2 B\{T_1 - \zeta_2 L_1\} + e_2 A^2 + f_2. \tag{3.10b}$$

The constants σ_1, σ_2, etc. appearing in (3.10) are given in table 3

We note that the nonlinearity of the above equations is quadratic. This is in contrast to the corresponding equations given by Schaeffer (1979) which describe the interaction of $2m$ and $2m+2$ cell modes for $m \geqslant 2$. The equations governing the interaction in the latter case have only cubic nonlinear terms.

M	X	Y
2	8·6	13·0
4	7·7	11·8
6	7·4	11·4
8	7·4	11·3
10	7·4	11·3

TABLE 1

n	ϵ_n	δ_n	γ_n
1	0·24	−0·00063	−0·000047
2	−0·24	0·001	0·00084

TABLE 2

n	σ_n	ζ_n	e_n	f_n
1	0·0026	2400	0·0002	31
2	0·0042	−3000·0	0·0035	−47

TABLE 3

Solution of the amplitude equations

We first consider the solution of (3.10) obtained by setting $f_1 = f_2 = 0$ in which case there is no motion forced by the end-wall boundary conditions. The equilibrium solutions of (3.10) are then given by

1. $A = B = 0$,
2. $B = -\sigma_1\{T_1 - \zeta_1 L_1\} e_1^{-1}, \quad A = \pm\{[T_1 - \zeta_1 L_1][T_1 - \zeta_2 L_1]\sigma_1\sigma_2/e_1 e_2\}^{\frac{1}{2}}$
3. $A = 0$, B undefined, $T_1 = \zeta_2 L_1$.

The first solution exists for all values of T_1 and L_1 whilst the second mixed mode solution exists only when $(T_1 - \zeta_1 L_1)(T_2 - \zeta_1 L_1) \geqslant 0$. Thus, for a given value of L_1 the latter mode exists if the Taylor number is less than the minimum or greater than the maximum of the critical Taylor numbers for the 2-cell and 4-cell modes. In addition to the solutions 1 and 2, there is a further solution given by 3. The stability of the equilibrium solutions shown above can be determined in the usual way by determining the growth rates of small perturbations to these solutions. The results obtained are shown in figure 3 and 4 for $L_1 > 0$ and $L_1 < 0$. We see that the zero solution is stable only for $T_1 < \min(\zeta_1 L_1, \zeta_2 L_1)$ whilst the mixed mode solution 2 is always unstable. The solution 3 is unstable for $B > \sigma_1 e_1^{-1}\{\zeta_1 L_1 - \zeta_2 L_1\}$ and neutrally stable for

$$B \leqslant \sigma_1 e_1^{-1}\{\zeta_1 L_1 - \zeta_2 L_1\}.$$

We now return to the case when f_1 and f_2 have the values given in table 3. If $T_1 \neq \zeta_2 L_1$ we can then show that the equilibrium solutions of (3.10) satisfy

$$e_1 e_2 A^3 + A[e_1 f_2 - \sigma_1\sigma_2(T_1 - \zeta_1 L_1)(T_1 - \zeta_2 L_1)] - f_1\sigma_2[T_1 - \zeta_2 L_1] = 0, \qquad (3.11a)$$

$$B = -[f_2 + e_2 A^2]/[\sigma_2(T_1 - \zeta_2 L_1)], \qquad (3.11b)$$

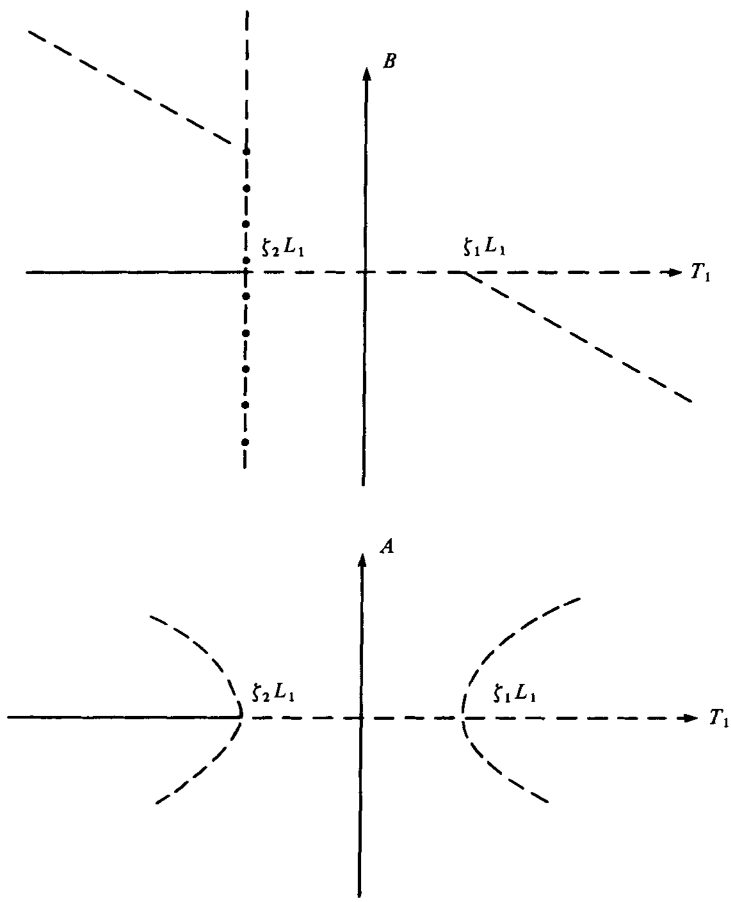

FIGURE 3. The solutions of (3.10) for $L_1 > 0, f_1 = f_2 = 0$. ——, stable solutions; $\cdot - \cdot - \cdot$, neutrally stable solutions; – – –, unstable solutions.

so that, depending on T_1 and L_1, either one or three equilibrium solutions exist. The nature of the solutions given by (3.11*a*, *b*) is shown in figures 5 and 6 for $L_1 > 0$ and $L_1 < 0$ respectively.

We first discuss the solutions shown in figure 5 which corresponds to the case when the 4-cell mode is the most dangerous mode of linear theory. The only stable branch is I which represents the smoothly developing primary flow. For all values of T_1 on I we have $A > 0$ and $B < 0$ so that the flow at the ends of the cylinders is always inwards as reported by Benjamin for the primary flow. However when $T_1 \to \zeta_2 L_1$ we see that the flows corresponding to I and IV become unbounded and a more careful examination of (3.10) shows that for $T_1 \sim \zeta_2 L_1$,

$$B \sim \frac{-f_2}{\sigma_2 [T_1 - \zeta_2 L_1]}, \quad A \sim \sigma_2 f_1 e_1^{-1} f_2^{-1} [T_1 - \zeta_2 L_1], \tag{3.12a, b}$$

and the above solution is unstable for $T_1 - \zeta_2 L_1 > 0$ and stable for $T_1 - \zeta_2 L_1 < 0$. This suggests that near $T_1 = \zeta_2 L_1$ there is an inner region in which the solution develops such that $B \gg A$. In fact in §4 we see that this region is of thickness $\alpha^{\frac{2}{3}}$

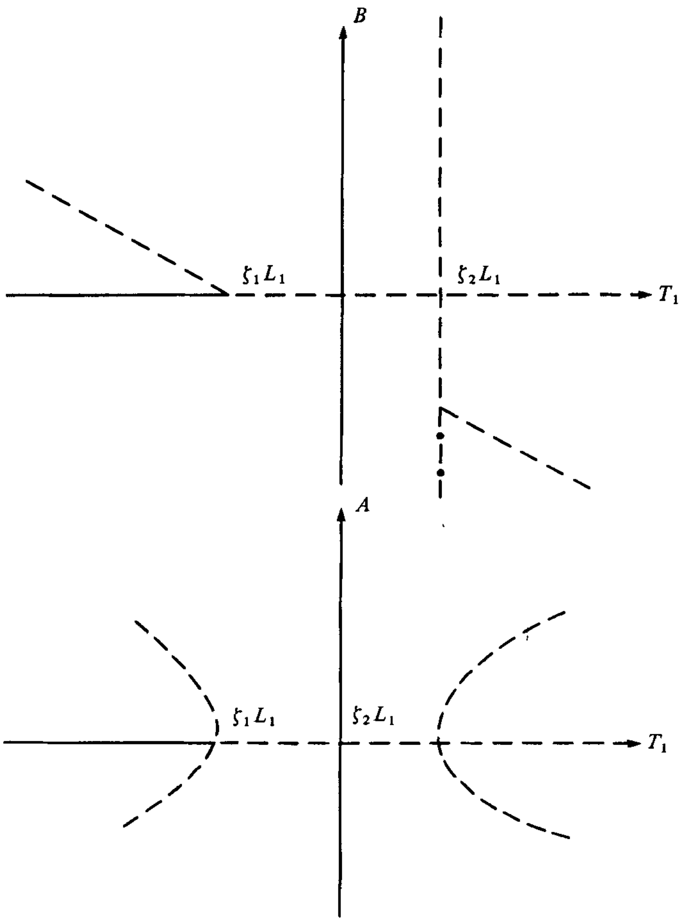

FIGURE 4. The same as figure 3 but with $L_1 < 0$.

and the stable resonating solution shown in figure 5 remains stable. We conclude that if $L > L^* + 0{\cdot}016\alpha^{\frac{1}{2}}$, the primary flow develops smoothly and is always a four-cell mode with flow inwards at the end walls.

Now suppose that we have $L_1 < 0$ so that according to linear theory the two-cell mode is the most dangerous.† We now see in figure 6 that the smoothly developing primary flow loses its stability when $T_1 = T_{c1}$. In this neighbourhood there is no other stable solution given by the present theory so we expect that, as Benjamin observed, the flow adjusts to some new secondary mode. However when $T_1 = T_{c2}$ the branch IV appears and is now a stable mode. This solution has the same asymptotic structure as given by (3.12) for $(T_1 - \zeta_2 L_2) < 0$ and small so that it corresponds to a four-cell mode again with radial inflow at the end walls. Clearly the branches I and IV will give similar flow patterns since they collapse into I of figure 5 when L_1 is increased through zero. We further note that since B changes sign on I for $T_1 < T_{c1}$, the primary flow can be either a two-cell or four-cell flow depending on T_1. For the values of f_1

† Note that this description also applies for $0 < L_1 < 0{\cdot}016$ where the four-cell mode is the most dangerous linear mode. Furthermore, when $L_1 = 0{\cdot}016$ the branches corresponding to I, II, IV and V meet and then $T_{c1} = T_{c2} = -60$.

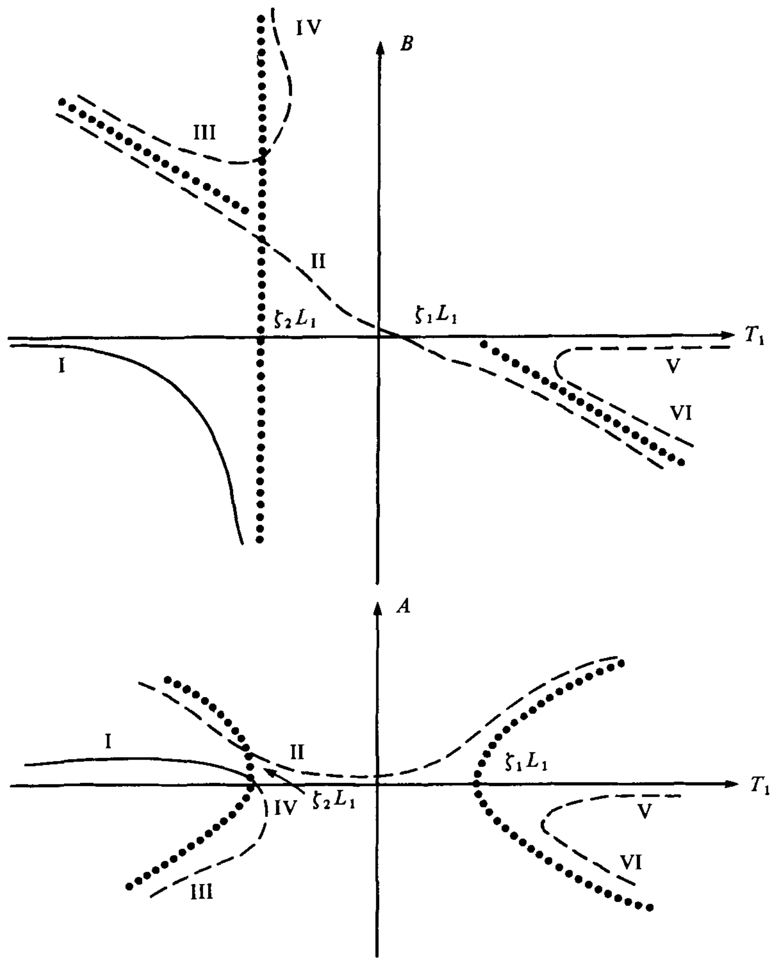

FIGURE 5. The solutions of (3.10) for $L_1 > 0{\cdot}016$. ——, stable solutions; – – –, unstable solutions; . . . ., the finite amplitude perfect solution.

and f_2 given in table 3, the values of T_{c1} and T_{c2} were calculated for $L_1 = -0{\cdot}1$ and we found that

$$T_{c1} = 2{\cdot}9\times10^2, \quad T_{c2} = 3{\cdot}0\times10^2.$$

Thus T_{c1} is decreased from the linear critical value $\zeta_1 L_1$ whilst T_{c2} is indistinguishable from the linear critical value $\zeta_2 L_1$.

We recall that the problem most relevant to experimental observations is that with $\alpha = 1$ in which case the flow satisfies the no-slip condition at the end walls. The theory which we have developed in this section is formally valid only in the limit $\alpha \to 0$. The radius of convergence of the expansions which we have developed can only be determined by proceeding to higher order in the expansions. Thus it is not clear that our results obtained for $\alpha \ll 1$ are relevant to the case $\alpha = 1$.

Nevertheless we now set $\alpha = 1$ and compare the predictions of our theory with the observations of Benjamin (1978). Suppose then that $L < L^*$ and we use the nonlinear theory above to predict the Taylor number at which the primary flow loses its stability together with the Taylor number at which the stable secondary four-cell mode emerges.

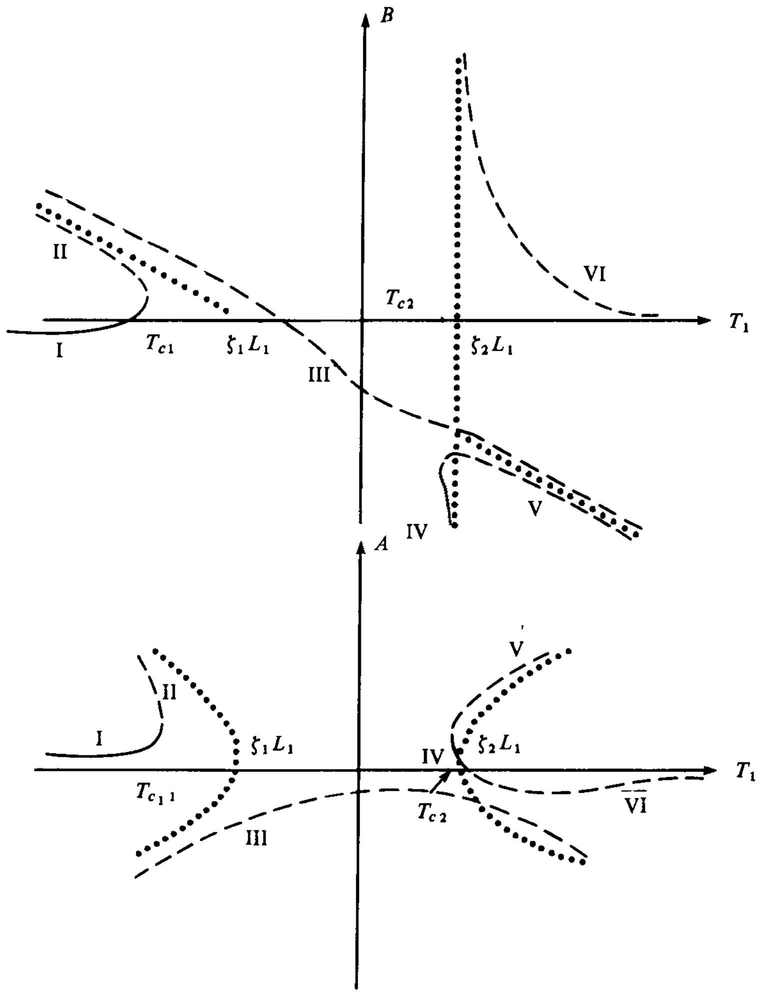

FIGURE 6. The same as figure 5 but with $L_1 < 0{\cdot}016$.

By varying L we can construct the curves C_a and C_b shown in figure 7 which, when crossed by increasing the Taylor number, predict the appearance of a stable four-cell mode and the disappearance of the primary flow respectively. We note the similarity of these curves with the curves BA and BC from figure 1 which we recall describe the same phenomena found experimentally by Benjamin. Since our theory is only valid for the small gap limit, a more accurate comparison with the experimental results is not possible. However, we shall return to this point in §5.

4. The nonlinear development of the resonating solutions

We have seen in §3 that if T_4^*, T_2^* represent the linear critical Taylor number for the four-cell and two-cell modes correct to order $\alpha^{\frac{1}{2}}$, then four- and two-cell modes of amplitude $\alpha/(T-T_4^*)$ and $T-T_4^*$ respectively develop for $T \sim T_4^*$. Thus the quadratic nonlinearity of the amplitude equations (3.10) is not sufficient to prevent the resonance due to the forcing of the four-cell mode by the end conditions. It is of interest

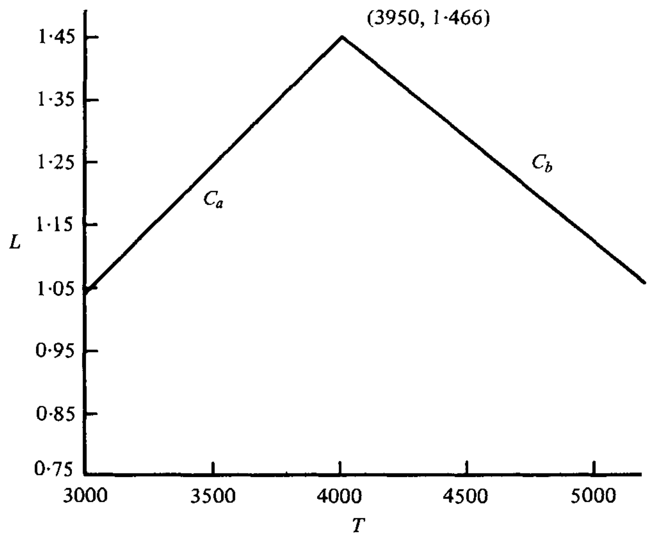

FIGURE 7. The curves C_a and C_b.

to note that, as found by Hall (1979) this resonance would be avoided if (3.10*b*) contained a quadratic term proportional to B^2. We shall now show how the resonating solution develops in an interval of width $\alpha^{\frac{2}{3}}$ about T_4^*.

The asymptotic structure given by (3.12) shows that $B \gg A$ in such an interval so we assume that the four-cell and two-cell modes have amplitude of order α^f and α^g respectively, where $0 < f < g$. In the absence of the two-cell mode we would expect that the resonance of the four-cell mode would be controlled by taking $f = 1/3$ (see, for example, Kelly & Pal 1976; Hall & Walton 1977). In this case the forced motion of order α becomes of order $\alpha^{\frac{1}{3}}$ in an interval of order $\alpha^{\frac{2}{3}}$ about T_4^*. Since $B \gg A$ we assume that the four-cell mode develops in this way and now determine the value of g which enables us to develop a consistent perturbations expansion.

We recall that a single interaction between the two-cell and four-cell modes reinforces the two-cell mode and that this mode is forced at order α by the end walls. In order to balance these effects we must therefore choose g such that $a^{f+g} \sim \alpha$. Hence we choose $g = \frac{2}{3}$ and expect a two-cell mode of amplitude $\alpha^{\frac{2}{3}}$. Thus we expand the Taylor number and length in the form

$$T = T^* + \zeta_2 L_1 \alpha^{\frac{1}{2}} + \overline{T}_1 \alpha^{\frac{2}{3}} + \ldots, \tag{4.1a}$$

$$L = L^* + L_1 \alpha^{\frac{1}{2}}. \tag{4.1b}$$

We have chosen the order $\alpha^{\frac{1}{2}}$ coefficient in (4.1*a*) to be that corresponding to the order $\alpha^{\frac{1}{2}}$ term in the asymptotic expansion of T_4^* in powers of $\alpha^{\frac{1}{2}}$. Thus we seek finite amplitude solutions in a region of thickness $\alpha^{\frac{2}{3}}$ about the critical Taylor number for the four-cell mode. Without any loss of generality we assume that all the coefficients except the first two in (4.1*b*) are zero. Thus for a given value of (L, α) the coefficient L_1 is determined by (4.1*b*).

We now define the three time scales τ_1, τ_2, τ_3 by

$$\tau_1 = \alpha^{\frac{1}{3}} t, \quad \tau_2 = \alpha^{\frac{1}{2}} t, \quad \tau_3 = \alpha^{\frac{2}{3}} t. \tag{4.2a,b,c}$$

The first time scale is the one on which the two-cell flow develops due to its interaction with the four-cell mode. The other two time scales are needed because, on the basis of linear theory, the two-cell and four-cell modes have growth rates of order $\alpha^{\frac{1}{2}}$ and $\alpha^{\frac{2}{3}}$ respectively. However, for the sake of simplifying the analysis, we shall for the moment seek only steady equilibrium solutions of the equations of motion. Some discussion of the effect of time dependent perturbations to these solutions will follow later in this section.

We expand $\mathbf{u} = (u, v, w)$ in the form

$$\mathbf{u} = \alpha^{\frac{1}{3}} \sum_{n=0}^{\infty} \alpha^{\frac{1}{3}n}\, \mathbf{U}_n(x, z)$$

and determine the functions $\mathbf{U}_n$ in the manner described in §3. If we substitute for $\mathbf{u}$ from above into (2.8) and use (4.1), we can show that the order $\alpha^{\frac{1}{3}}$, $\alpha^{\frac{1}{2}}$ solutions satisfying the required end conditions are

$$\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix} = B \begin{pmatrix} \cos 2k\phi u_2 \\ \cos 2k\phi v_2 \\ \dfrac{-\sin 2k\phi}{2k} u_2' \end{pmatrix},$$

$$\begin{pmatrix} U_1 \\ V_1 \\ W_1 \end{pmatrix} = C \begin{pmatrix} \cos 2k\phi u^2 \\ \cos 2k\phi v_2 \\ \dfrac{-\sin 2k\phi}{2k} u_2' \end{pmatrix}, \qquad (4.3a,b)$$

where (u_2, v_2) is the eigenfunction pair of the linear problem with wavenumber $2k$ and B and C are amplitude constants to be determined. We note that at this stage we have not yet allowed for the existence of a two-cell mode since, as argued earlier, this mode is of order $\alpha^{\frac{2}{3}}$. At order $a^{\frac{2}{3}}$ we find that $\mathbf{U}_2$ can be written in the form

$$\mathbf{U}_2 = B^2\mathbf{U}_{22}(x, \phi) + A \begin{pmatrix} \cos k\phi u_1 \\ \cos k\phi v_1 \\ \dfrac{-\sin k\phi}{k} u_1' \end{pmatrix}. \qquad (4.4)$$

Here (u_1, v_1) is the eigenfunction pair of the linear problem with wavenumber k whilst A is another amplitude constant to be determined. The function $\mathbf{U}_{22}$ comprises first harmonic terms corresponding to the wavenumber $2k$ together with mean flow terms. These terms automatically satisfy the end conditions and their precise form is not essential to the following analysis.

At order α we find that we obtain an inhomogeneous partial differential system which only has a solution if A and B satisfy the equations

$$0 = e_1 AB + f_1, \quad 0 = \sigma_2 \overline{T}_1 B - \lambda B^3 + f_2, \qquad (4.5a,b)$$

where the constants e_1, f_1, σ_2, f_2 are as given in table 3. The coefficient λ is determined by the integral conditions involving the order $\alpha^{\frac{2}{3}}$ function $\mathbf{U}_{22}$ together with (u_2, v_2) and its adjoint (u_2^+, v_2^+). In fact λ is identical to the coefficient of the cubic term in the corresponding amplitude equation of the infinite monochromatic problem with

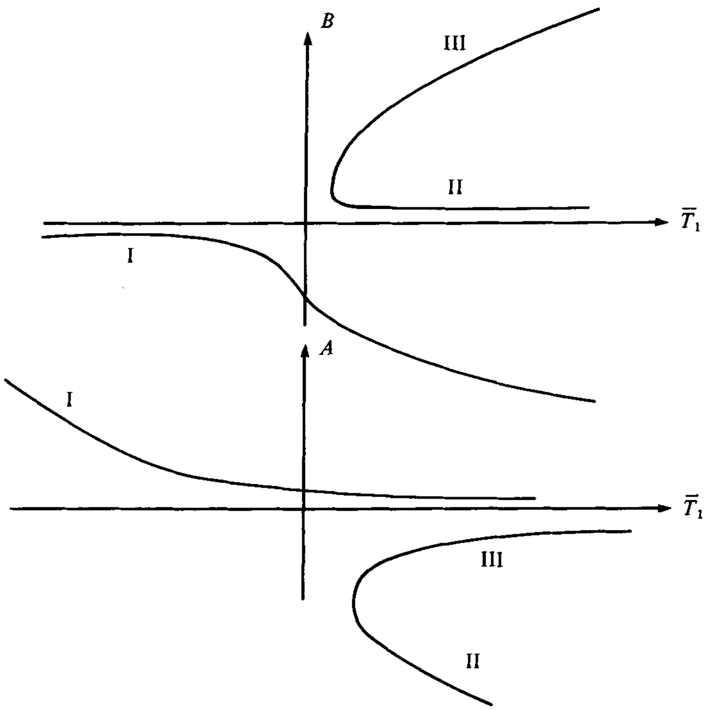

FIGURE 8. The solutions of (4.5).

wavenumber $2k$. It is well known that bifurcation is supercritical for the infinite problem so we can assume that $\lambda > 0$.

We now see that the amplitude of the four-cell mode is independent of the amplitude of the two-cell mode. Thus at any value of $\overline{T}_1$ we solve (4.5*b*) for B and then A is given by (4.5*a*). The solutions of (4.5) are illustrated in figure 8. For large values of $|\overline{T}_1|$ the asymptotic behaviour of A and B along I and II is

$$B \sim -f_2/\sigma_2\overline{T}_1, \quad A \sim f_1\sigma_2\overline{T}_1/e_1 f_2. \tag{4.6}$$

If we replace $\overline{T}_1$ above by $(T-T_4^*)/\alpha^{\frac{2}{3}}$ and $\overline{T}_1$ by $(T-T^*)/\alpha$ in (3.12) then, after noting that A and B are scaled on $\alpha^{\frac{1}{2}}$ in §3 whilst they are scaled on $\alpha^{\frac{2}{3}}$, $\alpha^{\frac{1}{3}}$ respectively here, we see that (4.6) matches with (3.12). Thus the stable resonating solutions of figures 4 and 5 develop smoothly along I whilst the unstable ones develop along II. For large positive values of T_1 the branch I has the asymptotic behaviour

$$B \sim -\left(\frac{\sigma_2\overline{T}_1}{\lambda}\right)^{\frac{1}{2}}, \quad A \sim \frac{f_1}{e_1}\left(\frac{\sigma_2\overline{T}_1}{\lambda}\right)^{-\frac{1}{2}}. \tag{4.7}$$

Thus the four-cell mode grows in amplitude and asymptotes to the equilibrium amplitude solution of the infinite problem at the same wavenumber. We see that at the same time the two-cell mode becomes even smaller in amplitude. Indeed the asymptotic form (4.7) shows that for $T-T_4^*$ positive and satisfying $\alpha^{\frac{2}{3}} \ll T-T_4^* \ll 1$ the four-cell and two-cell modes have amplitudes $(T-T_4^*)^{\frac{1}{2}}$ and $\alpha/(T-T_4^*)$ respectively. In particular if $T-T_4^* \sim O(\alpha^{\frac{1}{2}})$ there is an equilibrium configuration with two-cell and four-cell modes of respective amplitudes $\alpha^{\frac{1}{2}}$ and $\alpha^{\frac{1}{4}}$. We recall that the unstable solutions determined in §3 in this region had amplitude of order $\alpha^{\frac{1}{2}}$. We now turn to the question of the stability of the solutions shown in figure 8.

Suppose that we now allow the amplitude constants A, B and C to depend on the time variables τ_1, τ_2, τ_3 defined earlier by (4.2). It is easily shown that B can depend on τ_3 only and then (4.5b) becomes

$$dB/d\tau_3 = \sigma_2 \overline{T}_1 B - \lambda B^2 + f_2.$$

If we perturb the equilibrium flow by a small amount $be^{\sigma\tau_3}$ then we find that the growth rate σ of this perturbation is given by

$$\sigma = (\sigma_2 \overline{T}_1 - 2\lambda B^2)$$

where B is the equilibrium solution determined by (4.5b). It follows that the solutions represented by I and III in figure 8 are stable whilst II corresponds to an unstable solution.

However if we allow A to depend on τ_1, τ_2, τ_3 then (4.5a) becomes

$$\partial A/\partial \tau_1 = e_1 AB + f_1.$$

Thus if (A, B) is perturbed by a small amount $(ae^{\sigma\tau_1}, 0)$ from its equilibrium value given by solving (4.5) it follows that σ is given by

$$\sigma = e_1 B,$$

so that the solutions represented by II and I in figure 8 are respectively unstable and stable to such perturbation. Thus it follows that only I can possibly correspond to stable equilibrium flows. We can only show the stability of I by investigating the stability of A, B and in fact C and the other higher-order amplitude functions to perturbations dependent on τ_1, τ_2, τ_3. Such a procedure leads to an infinite set of coupled linear partial differential equations to determine the stability of the equilibrium flows. We do not pursue such a formidable procedure here but, since I matches onto the stable solutions of §3 and no secondary bifurcations occur, we can be reasonably certain that I is stable for all $\overline{T}_1$.

The above discussion enables us to give the following picture of the development of the Taylor vortex flow in cylinders of length $O(L^* + \alpha^{\frac{1}{2}} L_1)$ when the Taylor number is increased.

Suppose firstly that L_1 is positive so that the four-cell mode is the most dangerous linear disturbance. If $T \ll T^*$ then there is a weak circulatory flow with amplitude α. However when T is increased further resonance occurs for $T < T_4^*$ with $T - T_4^* \sim O(\alpha^{\frac{1}{2}})$. The flow now consists of two-cell and four-cell modes of amplitude $\alpha^{\frac{1}{2}}$. This flow has radial inflow at the ends and is the stable primary flow. However the four-cell mode becomes dominant when T is increased further. Thus in a region of order $\alpha^{\frac{2}{3}}$ about T_4^* the four-cell mode is of order $\alpha^{\frac{1}{3}}$ whilst the two-cell mode is of order $\alpha^{\frac{2}{3}}$. Moreover when T is increased further such that $T > T_4^*$ with $\alpha^{\frac{2}{3}} \ll T - T_4^* \ll 1$ the two-cell and four-cell modes have amplitude of order $(T - T_4^*)^{\frac{1}{2}}$ and $\alpha/(T - T_4^*)^{\frac{1}{2}}$. At this stage the four-cell mode has amplitude apparently independent of end effects. However, the end effects have selected to which solution of the infinite problem the primary fluid evolves in this Taylor number regime.

Now suppose that the value of L_1 chosen is negative so that $T_2^* < T_4^*$. The description given above remains valid until $T < T_2^*$ with $T - T_2^* \sim O(\alpha^{\frac{1}{2}})$. The smoothly developing primary flow now loses its stability and there is no other stable solution predicted by our theory. However, when T is increased further such that $T < T_4^*$ with $T - T_4^* \sim O(\alpha^{\frac{1}{3}})$ a stable secondary mode appears. The development of this mode in the neighborhood of $T = T_4^*$ is then identical to that described above for the primary flow. Thus in the regime $T > T_4^*$ with $\alpha^{\frac{2}{3}} \ll T - T_4^* \ll 1$ the flow is dominated by a four-cell mode of amplitude $(T - T_4^*)^{\frac{1}{2}}$. Furthermore this flow is to first-order indistinguishable from that to which the primary flow evolves for L_1 positive.

5. Weakly nonlinear theory for $(T - T^*) \sim (L - L^*) \sim 0(1)$ and the finite gap problem

In §§3, 4 we have determined the nonlinear development of the circulatory flows driven by the end-wall conditions when the length of the cylinders is close to the critical length $2L^*$. We shall now show how we obtain the corresponding development when $L - L^* \sim O(1)$. For the sake of definiteness we assume that L is such that $L < L^*$ so that $\hat{T}_2(L) < \hat{T}_4(L)$ in which case the two-cell mode is the most dangerous. Suppose further that K is the wavenumber corresponding to the point $(L, \hat{T}_2(L))$ on the neutral curve of the two-cell mode. It follows that

$$K = L/\pi$$

and, depending on L, K can be either of the two real positive wavenumbers k_1 and k_2 which are eigenvalues of (2.11). However we order the wavenumbers $\{k_n\}$ such that $k_1 = K$ and k_2 is the other positive wavenumber.

If T is an amount $O(1)$ below $\hat{T}_2(L)$ the forced motion is $O(\alpha)$ and can be solved by an eigenfunction expansion in terms of the eigenfunctions of (2.11). However when $T \sim \hat{T}_2$ the part of the solution arising from the eigenfunction with wavenumber K resonates and is then or order $\alpha/(T - \hat{T}_2)$. This resonating solution can be controlled in the usual way by balancing this motion with a finite amplitude disturbance of amplitude $(T - T_2)^{\frac{1}{2}}$. This suggests that we expand the Taylor number in the form

$$T = \hat{T}_2(L) + \alpha^{\frac{2}{3}} T_1 + \dots, \tag{5.1}$$

and we then expand $\mathbf{u}$ in the form

$$\mathbf{u} = \alpha^{\frac{1}{3}} \mathbf{U}_0 + \alpha^{\frac{2}{3}} \mathbf{U}_1 + \mathbf{U}_2 + \dots. \tag{5.2}$$

We further define a slow time variable τ by

$$\tau = \alpha^{\frac{2}{3}} \tau^*, \tag{5.3}$$

and we then substitute for T and $\mathbf{u}$ from (5.1), (5.2) into (2.8) and replace $\partial/\partial\tau^*$ by $\alpha^{\frac{2}{3}} \partial/\partial\tau$. If we equate terms of order $\alpha^{\frac{1}{3}}$ we can show that $\mathbf{U}_0$ is given by

$$\mathbf{U}_0 = X(\tau) \begin{pmatrix} \cos K\phi U \\ \cos K\phi V \\ \dfrac{-\sin K\phi}{K} U' \end{pmatrix}, \tag{5.4}$$

where $X(\tau)$ is an amplitude function to be determined and (U, V) is the eigenfunction pair corresponding to the wavenumber K. At order $\alpha^{\frac{3}{4}}$ we obtain a partial differential system for $\mathbf{U}_1$ which we solve to give

$$\mathbf{U}_1 = X^2(\tau)\begin{pmatrix}\cos 2K\phi U_{11} \\ \cos 2K\phi V_{11} \\ \dfrac{-\sin 2K\phi}{2K} U'_{11}\end{pmatrix} + X^2(\tau)\begin{pmatrix}0 \\ V_{10} \\ 0\end{pmatrix}. \tag{5.5}$$

Here the first harmonic function pair (U_{11}, V_{11}) satisfies an inhomogeneous form of (2.11) with $k_l = 2\pi$ whilst V_{10} is the usual mean flow correction. The solution (5.5) automatically satisfies the required end conditions since $KL = \pi$. Indeed at this stage the solution is identical to that which would be obtained for the infinite problem with axial wavenumber K and T given by (5.1).

At order α we obtain a partial eifferential system for $\mathbf{U}_2$ and if this is solved such that the side-wall conditions are satisfied we obtain.

$$\mathbf{U}_2 = -\left(\epsilon_1\frac{dX}{d\tau} + \delta_1 T_1 X + \theta_1 X^3\right)\begin{pmatrix}\phi\sin K\phi U \\ \phi\sin K\phi V \\ \left[\dfrac{K\phi\cos K\phi - \sin K\phi}{K^2}\right] U'\end{pmatrix} + \sum_{n=2}^{\infty}\mu_n\begin{pmatrix}\cos k_n\phi u_n \\ \cos k_n\phi v_n \\ \dfrac{-\sin k_n\phi}{k_n} u'_n\end{pmatrix}$$
$$+\{\text{terms proportional to } \sin nk\phi,\ \cos nk\phi,\ n = 1, 3\} \tag{5.6}$$

where ϵ_1, δ_1 are as defined by (3.6), (3.7) with $k_1 = K$. The constant θ_1 is determined by integral conditions involving (U_{11}, V_{11}), V_{10} and the eigenfunction pair (U, V) together with its adjoint. It suffices for our purposes to remark that θ_1 is always negative. The constants $\{\mu_n\}$, $n = 2, 3, \ldots$ and the coefficient $(\epsilon_1 dX/d\tau + \delta_1 T_1 + \theta_1 X^3)$ are then determined by requiring that the end conditions are satisfied. This can be done in exactly the same way described in § 3 and we obtain

$$\epsilon_1 dX/d\tau + \delta_1 T_1 X + \theta_1 X^3 = \eta_1, \tag{5.7}$$

where η_1 is a constant which depends on $T_2(L)$. The solutions of this amplitude equation are similar to those given earlier for $B(\tau)$ in § 4 and shown in figure 8. The primary solution is always stable and for $T_1 \gg 0$ the resonating solution has amplitude given by $\pm(-\delta_1 T_1/\theta_1)^{\frac{1}{2}}$ depending on whether η_1 is positive or negative. We further note that there is a stable secondary mode corresponding to II in figure 8. This solution corresponds to a flow with radial flow at the ends in the opposite direction to that corresponding to the primary flow. However we have shown that if the Taylor number is increased slowly with $L - L^* \sim O(1)$ a unique stable primary flow with two or four cells develops depending on whether $L < L^*$ or $L > L^*$. This is precisely the result obtained by Benjamin. For certain values of L, Benjamin found 'abnormal' four-cell flows with radial outflow near the end walls. The secondary mode corresponding to II in figure 8 could represent such a flow depending on the sign of η_1. We have not calculated this constant as a function of L but certainly in the limit $L \to L^*$ we can show from §3 that the sign of η_1 leads to such a flow.

The results which we have obtained so far in this paper agree closely with those given by Benjamin. However a direct comparison with our theoretical predictions and the data shown in figure 1 is not possible since we have used the small gap approximation. However we now make an effort to remedy this situation and determine a theoretical prediction for the coordinates of the point B in figure 1.

In fact to order α^0 the point B corresponds to the length and Reynolds number at which the two-cell and four-cell modes are equally likely. We follow the notation of Roberts (1965) and define a Taylor number T' by

$$T' = \left(\frac{2\Omega R_0^2}{\nu}\frac{1}{1-\eta^2}\right)^2, \tag{5.7}$$

where η, given by

$$\eta = \frac{R_0}{R_0+d}, \tag{5.8}$$

is the ratio of the radii of the cylinders. A Reynolds number R for the flow is defined by

$$R = \Omega R_0^2/\nu, \tag{5.9}$$

and we can show from (5.7), (5.9) that

$$R = T'^{\frac{1}{2}}.\tfrac{1}{2}(1-\eta^2). \tag{5.10}$$

If the cylinders are taken to be of length $2Ld$ then the point B of figure 1 corresponds to $R = 123$ and $L = 1{\cdot}86$. The apparatus used in the experiments of Benjamin corresponded to $\eta = 0{\cdot}615$. The eigenvalue problem for the wavenumbers a' of linear perturbations to the circumferential flow of the wide gap problem has been given by Roberts (1965) and is

$$\left.\begin{aligned} &(DD^*-a'^2)^2 u = -a^2 T'\left(\frac{1}{r^2}-1\right)v,\\ &(DD^*-a'^2)v = u,\\ &u = v = D^*u = 0, \quad r = \eta, 1. \end{aligned}\right\} \tag{5.11}$$

Here r is a radial variable scaled on (R_0+d) and the operators D, D^* are defined by

$$D = \frac{d}{dr}, \quad D^* = \frac{d}{dr}+\frac{1}{r}.$$

The axial wavenumber a' has been scaled on $1/(R_0+d)$ and must be determined numerically. The point B of figure 1 corresponds to Taylor number T' at which the two real wavenumbers a'_1 and a'_2 are such that

$$a'_1/a'_2 = \tfrac{1}{2},$$

and a'_1 and L are then related by

$$L = \frac{\pi}{a'_1(1-\eta)}.$$

The values of a'_1, L, and R at which this occurs were determined numerically for $\eta = 0{\cdot}615$ which we recall corresponds to the experiments of Benjamin. We obtained

$R = 126$ and $L = 1{\cdot}44$. Thus the theroretical prediction for the Reynolds number corresponding to B in figure 1 is extremely good, but the predicted value of L is not so good. We presume that the value of L does not agree so well because of the boundary layer structure which exists at the ends of the cylinders in the experimental configuration. However it is possible that the order $\alpha^{\frac{1}{4}}$ corrections to the co-ordinates of B might lead to better agreement between theory and experiment.

6. Conclusion

The model problem suggested by Schaeffer is indeed able to predict some of the results obtained by Benjamin. However the analysis given by Schaeffer is not relevant to the experimental situation which he is attempting to describe. It is of interest to note that if we had investigated the interaction problems appropriate to a neighbourhood of the point of intersection of the neutral curves of the $2m$ and $2m+2$ cell modes with $m \geqslant 2$ then the amplitude equations which determine the possible flows would have only cubic nonlinear terms. It is known that such amplitude equations can lead to hysteresis phenomena (see for example Hall & Walton 1979). Such a result was found by Schaeffer in his work if he assumed that the coefficients had certain properties but these coefficients were not evaluated by him. Moreover it is not known whether an experimental investigation appropriate to this case would lead to results similar to those shown in figure 1.

The development of the forced motion described in §§3 and 4 is not peculiar to the two-cell and four-cell interaction problem. The development of the forced motion which we have described in §4 is also relevant to the interaction problem between m and $2m$ cell flows for $m = 1, 3, 4, 5, \ldots$. However no experimental results are available for these cases and in fact it is likely that the only case of physical interest is when $m = 1$. This seems to be a reasonable assumption in view of the fact that at the Taylor number and lengths appropriate to the $m \geqslant 3$ cases other more unstable modes exist. We expect that the results of §§3 and 4 are relevant to other stability problems when two linear modes with wavenumbers having ratio $\frac{1}{2}$ become unstable at nearly the same Reynolds number.

The major deficiency of the present paper is its inability to predict the curve I of figure 1. We recall that when this curve is crossed a stable secondary flow with two cells is possible. There appears to be no suitable scaling leading to such a flow for values of L such that $(L-L^*) \ll 1$ whilst for $L > L^*$ and $(L-L^*) = O(1)$ the critical Taylor numbers for the two-cell and the four-cell flows differ by an $O(1)$ amount. In the latter case the interaction problem cannot be studied by perturbation means and is a numerical problem. Thus we feel that the curve I can only be determined by numerical means.

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